Representation of Groups

Mamoru Tanaka
mamoru.tanaka@wpi-aimr.tohoku.ac.jp
AIMR, Tohoku University, Japan

1 Purpose

In physics, group is an expression of the symmetry of a structure. Such groups are represented as a subgroup of the unitary group which is compatible with Hamiltonian on a Hilbert space. Under the represented group translation, the energy of the wave functions (in the Hilbert space) are invariant. We investigate properties of other representations of groups.

2 Group (Symmetry of Structure)

Mathematically, a *group* is a set G with product $G \times G \to G$; $(g_1, g_2) \mapsto g_1 g_2$ satisfying

(i) $g_1(g_2 g_3) = (g_1 g_2) g_3$ for all $g_1, g_2, g_3 \in G$;

(ii) there is $e \in G$ such that $ge = eg = g$ for all $g \in G$;

(iii) for any $g \in G$ there is $g^{-1} \in G$ such that $gg^{-1} = e$.

The following are well-known examples of groups in physics.

- Rotation group : the symmetry of a sphere
- Space group : the symmetry of a crystal configuration
- Poincare group : the symmetry of a Minkowski spacetime
- $\{+1, -1\}$: the symmetry under a time reversal
- Gauge group : the symmetry of a gauge
- Unitary group : the symmetry of a Hilbert space

A group is said to be *finitely generated* if there is a finite set $S \subset G$ such that any element of G is a product of elements in S. For example, $\{+1, -1\}$ and space groups are finitely generated.

A map π from a group G to a group G' is called a *homomorphism* if $\pi(g_1 g_2) = \pi(g_1)\pi(g_2)$ for all $g_1, g_2 \in G$.

3 Representation

Let M be a compact manifold. For example, M is a square or a cube with periodic boundary condition.

A *Hilbert space* is a function space on M defined as

$$H = L^2(M) := \{f : M \to \mathbb{C} \mid \int_M |f(x)|^2 d\mu(x) < \infty\}$$

with the inner product and the norm for $f_1, f_2, f \in H$

$$\langle f_1, f_2 \rangle := \int_M f_1(x)\overline{f_2(x)} d\mu(x), \quad \|f\|_2 := \left(\int_M |f(x)|^2 d\mu(x) \right)^{1/2}.$$

The *unitary group* $\mathcal{U}(H)$ of the Hilbert space H is the group of all invertible linear operators $U : H \to H$ such that

$$\langle Uf_1, Uf_2 \rangle = \langle f_1, f_2 \rangle$$

for all $f_1, f_2 \in H$, this is equivalent to

$$\|Uf\|_2 = \|f\|_2$$

for all $f \in H$. A *unitary representation* of a group G is a homomorphism $\pi : G \to \mathcal{U}(H)$. This means G is represented as $\pi(G)$ in $\mathcal{U}(H)$, and $\pi(G)$-translation don't change the "energy" $\|f\|_2$ of a function $f \in H = L^2(M)$.

Let $\Phi : [0, \infty) \to [0, \infty)$ be a convex non-negative function with $\Phi(0) = 0$, $\Phi(t) > 0$ for $t > 0$. The *Orlicz space* is a function space on M defined as

$$L^\Phi(M) = \{f : M \to \mathbb{R} \mid \int_M \Phi\left(\frac{|f(x)|}{b}\right) d\mu(x) < \infty \text{ for some } b > 0\}$$

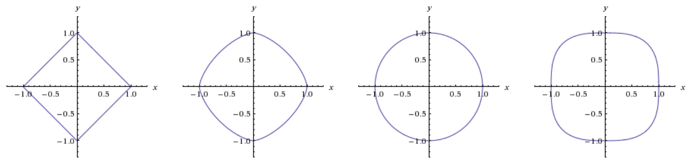
with the norm

$$\|f\|_\Phi := \inf\{b > 0 \mid \int_M \Phi\left(\frac{|f(x)|}{b}\right) d\mu(x) \le 1\}.$$

For example, if $\Phi_2(t) = t^2$, then $L^{\Phi_2}(M) = L^2(M)$ and $\|f\|_2 = \|f\|_{\Phi_2}$. For $1 \le p < \infty$, if $\Phi_p(t) = t^p$, then $L^{\Phi_p}(M)$ is written as $L^p(M)$ and satisfies

$$\|f\|_{\Phi_p} = \|f\|_p := \left(\int_M |f(x)|^p d\mu(x) \right)^{1/p}.$$

The unit circles of 2-dimensional Orlicz spaces for $\Phi(t) = t, t^{3/2}, t^2, t^3$ are



For $B = L^\Phi(M)$, the *linear isometric group* $O(B)$ is the group of all invertible linear operators $O : B \to B$ such that

$$\|Of\|_\Phi = \|f\|_\Phi$$

for all $f \in B$. A *linear isometric representation* of a group G is a homomorphism $\pi : G \to O(B)$. This means G is represented as $\pi(G)$ in $O(B)$, and $\pi(G)$-translation don't change the "Φ-energy" $\|f\|_\Phi$ of a function $f \in B$.

4 Property (T_B)

Let G be a finitely generated group with the generating set S, $B = L^\Phi(M)$, and π a linear isometric representation of G. We denote by $B^{\pi(G)}$ the set of the invariant functions, i.e., functions $f \in B$ such that $\pi(g)f = f$ for all $g \in G$. Then we can define the linear isometric representation $\pi' : G \to O(B/B^{\pi(G)})$, which doesn't have invariant functions. We say that π' *almost has invariant vectors* if there is a sequence $f_n \in B/B^{\pi(G)}$ $(n = 1, 2, \dots)$ such that

$$\lim_{n\to\infty} \max_{g \in S} \|\pi'(g)f_n - f_n\|_{B/B^{\pi(G)}} = 0.$$

The group G is said to have *property* (T_B) if for any linear isometric representation $\pi : G \to O(B)$, the induced representation $\pi' : G \to O(B/B^{\pi(G)})$ does not almost have invariant vectors. This means that if $f \in B$ is "orthogonal" to every invariant functions, then it is not even almost invariant.

Theorem ([BFGM07]). *Let G be a finitely generated group and $1 < p < \infty$. Then G has property $(T_{L^2(M)})$ if and only if it has property $(T_{L^p(M)})$.*

An feature of $L^p(M)$ $(1 < p < \infty)$ is uniform convexity and uniform smoothness. So we consider uniformly convex and uniformly smooth Orlicz spaces.

5 Result

Theorem ([T]). *Let G be a finitely generated group and $L^\Phi(M)$ a uniformly convex and uniformly smooth Orlicz space. Then G has property $(T_{L^2(M)})$ if and only if it has property $(T_{L^\Phi(M)})$.*

Remark. Uniform convexity and uniform smoothness of $L^\Phi(M)$ are written in terms of Φ.

References

[BFGM07] U. Bader, A. Furman, T. Gelander, N. Monod, Acta Math. 2007,

[T] M. Tanaka, in preparation.